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THE ONSET OF AUTO-OSCILLATIONS IN A FLUID

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The onset of auto-oscillations at transition of the Reynolds number (or any other parameter defining the steady motion of a viscous incompressible fluid) through its critical value is investigated.

Landau in [1] (see, also, [2, 3]) considered the onset of the periodic auto-oscillation mode to be the first stage of transition from a laminar to a turbulent flow of a fluid. His method, developed also by Meksyn, Stuart and Watson (see [4–7]), implies the knowledge of the eigenvectors of the linearized (with respect to the basic laminar mode at a given Reynolds number) Navier-Stokes operator to which (according to the linear theory) correspond increasing perturbations. A system of ordinary nonlinear differential equations is derived for the determination of the Fourier coefficients of the velocity field. The calculation of the right-hand sides of equations of this system is, however, somewhat involved. Owing to this, this method had not, so far, provided final results in specific cases, such as, for example, the Poiseuille flow in a channel. The Landau method is clearly more suitable for investigating the onset of a periodic mode rather than for the calculation of a stabilized one.

Here the onset of auto-oscillations is analyzed by the Liapunov-Schmidt method described in [8, 9]. The branching out of periodic solutions of systems of ordinary differential equations is considered in [10], where references to earlier works are cited. The generation of a cycle is considered in [10, 11] for a system of ordinary differential equations, while [12–14] deal with the special case of Galerkin equations approximating the Navier-Stokes system. Certain statements related to the complete Navier-Stokes equations are also formulated in [13, 14].

A comprehensive statement of the problem and basic definitions are given in Sect. 1; an a priori estimate of possible auto-oscillation modes is presented (Lemma 1.2), and it is shown that only the critical value of a parameter can be a point of branching out of the system (Lemma 1.3).

This is followed by the analysis of supplementary conditions for the actual generation of a cycle. Theorem 2.1, which is an analogy of Krasnosel'skii's theorem on bifurcation [15], is proved in Sect. 2. The existence of periodic auto-oscillatory motion under conditions of Theorem 2.1 is established by the analysis of linear equations only, independently of the form of nonlinear terms.

A more detailed analysis of generated cycles, of their number and analytic properties is given in Theorem 2.2 and related notes in terms of parameter γ_*.

Since the proofs of Theorems 2.1 and 2.2 are based on the most general properties of Navier-Stokes equations, these theorems can be readily extended to a wide class of ordinary differential equations in a Banach space (see Theorems 3.1 and 3.2 in Sect. 3) which comprise, in particular, problems involving equations of the parabolic kind, equations of convection, magnetohydrodynamics, etc.

1. Existence of auto-oscillations. Statement of the problem. Let a homogeneous viscous incompressible fluid fill the bounded region Ω of a three-dimensional Euclidean space (*). Let us assume that the vectors of mass forces and of velocity at the boundary S of region Ω are specified and, while being independent of time, depend on a certain parameter γ.

Let there exist a stationary solution of the Navier-Stokes equations $(\mathbf{a}(x, \gamma),\ p_0(x, \gamma))$ which we shall call in the following the fundamental solution.

We denote by γ_0 the critical value of parameter γ, when for $\gamma = \gamma_0$ the stability spectrum of the basic flow has a nonempty intersection with the imaginary axis.

If for $\gamma = \gamma_0$ the stability spectrum contains zero, the passage of parameter γ through the critical value γ_0 results, as a rule, in the branching out of new stationary modes (see [16–18]). However here we consider the case in which at $\gamma = \gamma_0$ the stability spectrum contains a pair of purely imaginary eigenvalues $\mp i\omega_0$ $(\omega_0 \neq 0)$. In this case the linearized system has a periodic solution, and it can be expected that for γ close to γ_0 there exists a periodic auto-oscillatory solution of nonlinear Navier-Stokes equations. The conditions for an actual occurrence of this are given later.

The stability spectrum of the auto-oscillatory mode for γ close to γ_0 contains points $\sigma_{1,2}$ close to $\mp i\omega_0$ (these can be calculated by using series expansions of the perturbation theory). If these appear in the right-hand half-plane, the auto-oscillation is unstable. It is, also, unstable, when for $\gamma = \gamma_0$ the basic flow is unstable, and this manifests itself by the appearance in its stability spectrum of points of the right-hand half-plane. If, however, at $\gamma = \gamma_0$ all points of the basic flow stability spectrum, except $\mp i\omega_0$, lie within the left-hand half-plane and $\operatorname{Re}\sigma_{1,2} < 0$, the auto-oscillation mode is stable.

Let us assume in what follows that $S \in C^2$ and that the dependence of vector $\mathbf{a}$ on γ is analytic in the neighborhood of γ_0

$$\mathbf{a}(x, \gamma) = \sum_{k=0}^{\infty} \mathbf{a}_k(x)\,\delta^k, \qquad \delta = \gamma - \gamma_0 \tag{1.1}$$

and series (1.1) is convergent in $W_2^{(2)}$.

Assuming that for any solution of the Navier-Stokes equations $\mathbf{v}'$ and P' are expressed by

$$\mathbf{v}' = \mathbf{v} + \mathbf{a}, \qquad P' = q + P_0 \tag{1.2}$$

*) Two- and n-dimensional cases are treated similarly.

we obtain for perturbations the nonlinear equation

$$\frac{d\mathbf{v}}{dt} + A\mathbf{v} + \sum_{k=1}^{\infty} \delta^k B_k \mathbf{v} = -K\mathbf{v} \tag{1.3}$$

The following notation is used here:

$$\begin{gathered} K\mathbf{v} = K_{20}(\mathbf{v}, \mathbf{v}), \qquad K_{20}(\mathbf{u}, \mathbf{v}) = \Pi(\mathbf{u}, \nabla)\,\mathbf{v} \\ K_{20}{}^{\circ}(\mathbf{u}, \mathbf{v}) = K_{20}(\mathbf{u}, \mathbf{v}) + K_{20}(\mathbf{v}, \mathbf{u}), \qquad B_k\mathbf{v} = K_{20}{}^{\circ}(\mathbf{a}_k, \mathbf{v}) \quad (k = 0, 1, \ldots) \\ A = A_0 + B_0, \qquad A_0 = -\nu\Pi\Delta \end{gathered} \tag{1.4}$$

The operator Π is the orthogonal projector in L_2 into the subspace and $H = S_2$ is the closure in L_2 of the set of smooth solenoidal vectors which vanish in the neighborhood of the boundary of region Ω.

The unknown cyclic frequency of the sought periodic solution of Eq. (1.3) will be denoted by ω. Substituting $\omega t = \tau$, we reduce Eq. (1.3) to the form

$$\omega\frac{d\mathbf{v}}{d\tau} + A\mathbf{v} + \sum_{k=1}^{\infty} \delta^k B_k \mathbf{v} = -K\mathbf{v} \tag{1.5}$$

Let us assume that a unique eigenvector φ of operator A corresponds to eigenvalue $-i\omega_0$. Then the complex conjugate eigenvector φ^*

$$A\varphi + i\omega_0\varphi = 0, \qquad A\varphi^* - i\omega_0\varphi^* = 0 \tag{1.6}$$

corresponds to the eigenvalue $i\omega_0$.

We introduce operator A^* conjugate of operator A in H whose region of definition coincides with D_A, and

$$A^* = A_0 + B_0^*, \qquad B_0^*\mathbf{u} = -\Pi\left\{a_k\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\mathbf{e}_i\right\} \tag{1.7}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the coordinate unit vectors in R^3. For any $\mathbf{u}$, and $\mathbf{v} \in D_A$ we have the identity

$$(A\mathbf{u}, \mathbf{v})_H = (\mathbf{u}, A^*\mathbf{v})_H \tag{1.8}$$

The scalar product in H is of the form

$$(\mathbf{u}, \mathbf{v})_H = \int_{\Omega} \mathbf{u}\cdot\mathbf{v}^* dx \tag{1.9}$$

Operator A has a discrete spectrum whose resolvent is an absolutely continuous operator in the energy space H_1 of operator A_0; each generalized eigenvector φ belongs to D_{A}, and every eigenvalue of operator A is, also, the eigenvalue of the conjugate operator A^*. We denote the eigenvector of operator A^* corresponding to the eigennumber $i\omega_0$ by Φ. We have

$$A^*\Phi - i\omega_0\Phi = 0, \qquad A^*\Phi^* + i\omega_0\Phi^* = 0 \tag{1.10}$$

Let us assume that the eigenvalues $\mp i\omega_0$ are simple numbers; this implies not only the uniqueness of related eigenvectors but, also, that $(\varphi, \Phi)_H \neq 0$. Hence it can be assumed that condition

$$(\varphi, \Phi)_H = \int_{\Omega} \varphi\cdot\Phi^* dx = 1 \tag{1.11}$$

is satisfied.

We look for nontrivial 2π-periodic solutions of Eq. (1.5) in the Hilbert space H_2 of closure of the set of vector functions $\mathbf{v}(\tau) : \tau \in [0, 2\pi]$ such that $A\mathbf{v}(\tau)$ and

$d\mathbf{v}(\tau)/d\tau$ are strongly continuous along $[0, 2\pi]$ in the metric

$$(\mathbf{u}, \mathbf{v})_{H_2} = \int_0^{2\pi} \left[\omega_0^2 \left(\frac{d\mathbf{u}}{d\tau}, \frac{d\mathbf{v}}{d\tau}\right)_H + (A_0\mathbf{u}, A_0\mathbf{v})_H\right] d\tau \tag{1.12}$$

Let the number γ_0 denote the point of branching out of a cycle, when there exist the sequence $\gamma_n \to \gamma_0$ $(\delta_n = \gamma_n - \gamma_0 \to 0)$ and the related sequencies of numbers $\omega_n \neq 0$ and vector functions $v_n \in H_2$ with $\mathbf{v}_n \neq 0$ which are solutions of Eq. (1.5) for $\mathbf{v}_n \to 0$ in H_2. Let us assume that a normal cycle is branching out, when there exist one-parameter sets ω_γ, v_γ: $\mathbf{v}_\gamma \in H_2$ for $\mathbf{v}_\gamma \neq 0$ $(\gamma \neq \gamma_0)$, continuous with respect to γ and satisfying Eq. (1.5), when γ passes through a certain interval J whose limit point is γ_0 and, when $\omega_\gamma \to \omega_0 \neq 0$ with $v_\gamma \to 0$ for $\gamma \to \gamma_0$. If $\omega_\gamma \to 0$ $(\omega_\gamma \to \infty)$ with all other conditions satisfied, the cycle will be called slow (fast). If the interval J can be chosen so as to comprise point γ_0, the cycle (whether normal, slow, or rapid) will be called two-sided. Otherwise the cycle will be called one-sided (*).

In what follows we examine the conditions which must be satisfied, if γ_0 is to be the branching out point of a cycle and, also, investigate the set of γ and ω for which Eq. (1.5) has a nontrivial solution.

We introduce operator $L : H_2 \to H' = L_2((0, 2\pi), H)$ on the assumption that for any vector function $\mathbf{u} \in H_2$

$$L\mathbf{u} \equiv \omega_0 \frac{d\mathbf{u}}{d\tau} + A\mathbf{u} \tag{1.13}$$

Lemma 1.1. For operator L to be invertible it is necessary and sufficient that points $in\omega_0$ $(n = 0, \mp 1, \ldots)$ do not appear in the spectrum of operator A.

Proof. It can be assumed that $\omega_0 > 0$ (in the contrary case this can be achieved by the substitution $\omega_0 \to -\omega_0$ and $\tau \to -\tau$). Let us consider equation

$$L\mathbf{u} \equiv \omega_0 \frac{d\mathbf{u}}{d\tau} + A\mathbf{u} = \mathbf{f} \tag{1.14}$$

on the assumption that $f \in H'$. We shall prove that the problem of finding a 2π-periodic solution of this equation reduces to a Fredholm equation of the second kind. For this we rewrite Eq. (1.14) in the form

$$L_0\mathbf{u} \equiv \omega_0 \frac{d\mathbf{u}}{d\tau} + A_0\mathbf{u} = f - B_0\mathbf{u} \tag{1.15}$$

It will be readily seen that operator $L_0 : H_2 \to H'$ is invertible. The inverse operator may be presented in various forms

$$u_0(t) = (L_0^{-1}\mathbf{f})(t) = \frac{1}{\omega_0} \int_{-\infty}^{t} \exp\left[-\frac{1}{\omega_0}(t-\tau) A_0\right] \mathbf{f}(\tau)\, d\tau =$$

$$= \sum_{n=-\infty}^{+\infty} e^{int} (in\omega_0 I + A_0)^{-1} \frac{1}{2\pi} \int_0^{2\pi} \mathbf{f}(\tau) e^{-in\tau}\, d\tau \tag{1.16}$$

$$u_0(t) = \sum_{k=1}^{\infty} \sum_{n=-\infty}^{+\infty} e^{int} \varphi_k \frac{1}{\lambda_k^2 + in\omega_0} \frac{1}{2\pi} \int_0^{2\pi} (\mathbf{f}(\tau), \varphi_k)\, e^{-in\tau}\, d\tau \tag{1.17}$$

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*) This definition extends to the case in which the transition of parameter γ through the critical value γ_0 results in the onset of cycles of the two or, even, all three kinds described above.

Here φ_k is the complete system of eigenvectors (self-conjugate, positive definite, and having an absolutely continuous inverse) of operator A_0 and λ_k^2 are the related eigenvalues $\lambda_k^2 \to \infty$ $(k \to \infty)$. According to the results in [20] we have

$$\varphi_k \in W_p^{(2)}(\Omega) \qquad \text{for} \quad p \geqslant 1$$

Inversion of operator L_0 reduces Eq. (1.15) to the equivalent form

$$\mathbf{u} + L_0^{-1}B_0\mathbf{u} = \mathbf{u}_0, \qquad \mathbf{u}_0 = L_0^{-1}\mathbf{f} \tag{1.18}$$

Let us prove that operator $L_0^{-1}B_0 : H_2 \to H_2$ is obsolutely continuous. In fact, it can be presented in the form $L_0^{-1}B_0J$, where J is the operator of imbedding of space H_2 into space H_1' of 2π-periodic vector functions from $L_2((0,\ 2\pi),\ H_1)$. For any vector function

$$u(t) = \sum_{k=1}^{\infty} \sum_{n=-\infty}^{+\infty} c_{nk} l^{int}$$

we have

$$\|\mathbf{u}\|_{H_2}^2 = 2\pi \sum_{n,k} (n^2\omega_0^2 + \lambda_k^4)\,|c_{nk}|^2, \quad \|\mathbf{u}\|_{H_1'}^2 = 2\pi \sum_{n,k} \lambda_k^2\,|c_{nk}|^2 \tag{1.19}$$

The criterion of compactness in l_2 implies thus that a sphere in space H_2 is a compact ellipsoid in H_1'. Hence operator $J : H_2 \to H_1'$ is absolutely continuous. Operator $B_0 :$ $: H_1' \to H'$ is bounded. Taking into consideration its definition (1.4) and the elementary inequality

$$\|\mathbf{u}\|_{H_1'}^2 \geqslant \lambda_1^2 \|\mathbf{u}\|_{H'}^2$$

we obtain

$$\|B_0\mathbf{u}\|_{H'} \leqslant c_0 \|\mathbf{u}\|_{H_1'}$$

$$c_0^2 = 2\max_{x\in\Omega} |a(x)|^2 + 2\frac{1}{\lambda_1^2} \int_\Omega (\mathrm{rot}\, a)^2\, dx \tag{1.20}$$

Thus operator $L_0^{-1}B_0 = L_0^{-1}B_0J$ is absolutely continuous in H_2. According to Fredholm's theory for Eq. (1.18) (or Eq. (1.14)) to be solvable it is necessary and sufficient that the related homogeneous equation has no trivial solutions. An expansion into Fourier series readily shows that the latter condition is the same as the condition of this Lemma, which completes the proof.

L e m m a 1. 2. The set of such ω which correspond to nontrivial 2π-periodic solutions of Eq. (1.5) from any sphere $\|\mathbf{v}\|_{H_1} \leqslant r$ is bounded by the number depending on r, $\|\mathbf{a}\|_{W_2^{(2)}}$ and on region Ω.

P r o o f. Let $\mathbf{v}$ be a nontrivial 2π-periodic solution of Eq. (1.5). We assume that

$$\mathbf{v} = \mathbf{v}^\circ + \mathbf{u}, \quad \mathbf{v}^\circ = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{v}(\tau)\, d\tau \tag{1.21}$$

The vector function $\mathbf{u}$ satisfies equation

$$\omega\,(d\mathbf{u}/d\tau) + A_0\mathbf{u} = -K(\mathbf{v}^\circ + \mathbf{u}) + K_0(\mathbf{v}^\circ + \mathbf{u}) - B\mathbf{u} \equiv g(\mathbf{u}, \mathbf{v}^\circ)$$

$$K_0(\mathbf{v}^\circ + \mathbf{u}) = \frac{1}{2\pi}\int_0^{2\pi} K(\mathbf{v}^\circ + \mathbf{u})\, d\tau, \quad B\mathbf{u} = \sum_{k=0}^{\infty} \delta^k B_k \mathbf{u} \tag{1.22}$$

From the definition (1.21) of the vector function $\mathbf{u}$ follows the estimate

$$\|\mathbf{u}\|_{H_1} \leqslant \|\mathbf{v}\|_{H_1}. \tag{1.23}$$

Integrating in both parts of Eq. (1.22) the scalar squares in H with respect to τ from 0 to 2π, we obtain

$$J_\omega \equiv \int_0^{2\pi} \left(\omega^2 \left\|\frac{d\mathbf{u}}{d\tau}\right\|_H^2 + \|A_0\mathbf{u}\|_H^2\right) d\tau = \int_0^{2\pi} \|g(\mathbf{u}, \mathbf{v}^\circ)\|_H^2 \, d\tau \tag{1.24}$$

Using inequality (1.21) and the elementary inequality

$$\left\|\sum_{k=1}^m f_k\right\|_{H'}^2 \leqslant m \sum_{k=1}^m \|f_k\|_{H'}^2 \tag{1.25}$$

for the estimate of the right-hand side of (1.24), we obtain

$$J_\omega \leqslant 5c_0^2 \|\mathbf{u}\|_{H_1'}^2 + 5 \int_0^{2\pi} d\tau \int_\Omega [\mathbf{v}^2 (\operatorname{rot} \mathbf{u})^2 + \mathbf{u}^2 (\operatorname{rot} \mathbf{v})^2 + \mathbf{u}^2 (\operatorname{rot} \mathbf{u})^2] \, dx \tag{1.26}$$

In what follows we shall need the imbedding inequalities

$$\max_{0 \leqslant \tau \leqslant 2\pi} \|\mathbf{v}(\tau)\|_{H_1} \leqslant c_1 \|\mathbf{v}\|_{H_2}, \qquad \|\mathbf{v}\|_{L_{10}(Q)} + \|\operatorname{rot} \mathbf{v}\|_{L_{10/3}(Q)} \leqslant c_2 \|\mathbf{v}\|_{H_2} \tag{1.27}$$
$$Q = \Omega \times [0, 2\pi]$$

where constants c_1 and c_2 depend only on region Ω. To prove the first of inequalities (1.27) we have to examine the identity

$$\frac{d}{d\tau} \|u\|_{H_1}^2 = \left\|\frac{du}{d\tau} + A_0 u\right\|_H^2 - \left\|\frac{du}{d\tau}\right\|_H^2 - \|A_0 u\|_{H'}^2, \qquad \nu = 1 \tag{1.28}$$

Using the inequality

$$\|A_0 u\|_H \geqslant \lambda_1 \|u\|_{H_1}$$

from (1.28) we obtain

$$\frac{d}{dt} \|\mathbf{u}\|_{H_1}{}^2 + \lambda_1{}^2 \|\mathbf{u}\|_{H_1}{}^2 \leqslant 2 \left(\left\|\frac{d\mathbf{u}}{dt}\right\|_H^2 + \|A_0\mathbf{u}\|_H{}^2\right) \equiv \varphi^2(t) \tag{1.29}$$

Multiplying both sides of this inequality by $e^{\nu\lambda_1^2 t}$ and integrating with respect to t from $-\infty$ to τ, we obtain

$$e^{\lambda_1^2 \tau} \|\mathbf{u}(\tau)\|_{H_1}{}^2 \leqslant 2 \int_{-\infty}^{\tau} e^{\nu\lambda_1^2 t} \varphi^2(t) \, dt \tag{1.30}$$

The right-hand part of (1.30) can be presented in the form

$$\sum_{n=0}^{\infty} \int_{\tau-2\pi(n+1)}^{\tau-2\pi n} e^{\lambda_1^2 t} \varphi^2(t) \, dt = \frac{e^{\lambda_1^2(\tau - 2\pi)}}{1 - e^{-2\pi\lambda_1^2}} \int_0^{2\pi} e^{\lambda_1^2 s} \varphi^2(\tau + s) \, ds \tag{1.31}$$

This yields directly the necessary inequality; constant c_1 can be taken as equal to $2(1 - e^{-2\pi\lambda_1^2})^{-1}$.

The second of inequalities (1.27) is derived in the manner given in [19].

We shall also require the multiplicative inequalities

$$\|\operatorname{rot} \mathbf{u}\|_{L_{5/2}(\Omega)} \leqslant C_3 \|\mathbf{u}\|_{H_1}^{7/10} \cdot \|A_0\mathbf{u}\|_H^{3/10}, \qquad \|\mathbf{u}\|_{L_5(\Omega)} \leqslant C_4 \|\mathbf{u}\|_H^{1/10} \cdot \|\mathbf{u}\|_{H_1}^{9/10} \tag{1.32}$$

Finally, we note the inequalities

$$\|\mathbf{u}\|_{H'}^2 \leqslant 1/\omega^2 J_\omega, \qquad \|\mathbf{u}\|_{H_1'}^2 \leqslant \frac{1}{\omega} J_\omega \tag{1.33}$$

valid for any vector function $\mathbf{u} \in H_2$; $u^\circ = 0$

To prove these it is sufficient to expand the vector function $\mathbf{u}$ into a Fourier series, use the Parceval equality (1.19), etc.

Let us revert now to relationship (1.26) and successively estimate the terms in its right-hand part.

Using the Hölder inequality together with the first of inequalities (1.32), we obtain

$$\int_0^{2\pi} d\tau \int_\Omega \mathbf{v}^2 (\operatorname{rot} \mathbf{u})^2\, dx \leqslant c_3^2 \| \mathbf{v} \|^2_{L_{10}(Q)} \left[\int_0^{2\pi} \| \mathbf{u} \|^{7/4}_{H_1} \cdot \| A_0 \mathbf{u} \|^{9/4}_{H}\, d\tau \right]^{4/5} \tag{1.34}$$

Using once again the Hölder inequality, we derive

$$\int_0^{2\pi} d\tau \int_\Omega \mathbf{v}^2 (\operatorname{rot} \mathbf{u})^2\, dx \leqslant c_3^2 \| \mathbf{v} \|^2_{L_{10}(Q)} \cdot \| A_0 \mathbf{u} \|^{9/5}_{L_2(Q)} \cdot \max_\tau \| \mathbf{u}(\tau) \|^{7/5}_{H_1} \cdot \| \mathbf{u} \|_{H_1}, \tag{1.35}$$

Finally, making use of inequalities (1.27) and (1.33) for the estimate of the right-hand side of (1.35), we obtain

$$\int_0^{2\pi} d\tau \int_\Omega \mathbf{v}^2 (\operatorname{rot} \mathbf{u})^2\, dx \leqslant \frac{c_5 r^2}{\sqrt{\omega}} J_\omega, \qquad c_5 = c_3^2 r_2^2 c_1^{9/5} \tag{1.36}$$

Inequalities

$$\int_0^{2\pi} d\tau \int_\Omega \mathbf{u}^2 (\operatorname{rot} \mathbf{v})^2\, dx \leqslant \frac{c_6 r^2}{\sqrt{\omega}} J_\omega, \qquad c_6 = c_4^2 c_2^2 c_1^{9/5} \tag{1.37}$$

$$\int_0^{2\pi} d\tau \int_\Omega \mathbf{u}^2 (\operatorname{rot} \mathbf{u})^2\, dx \leqslant \frac{c_5 r^2}{\sqrt{\omega}} J_\omega \tag{1.38}$$

are derived in a similar manner. Use was made of (1.23) in the derivation of inequality (1.38).

From (1.26), (1.33) and (1.36)–(1.38) we obtain

$$1 \leqslant \frac{5c_0^2}{\omega} + \frac{c_7 r^2}{\sqrt{\omega}} \tag{1.39}$$

This yields the estimate of ω

$$\omega \leqslant {}^1/_4 \left(c_7 r^2 + \sqrt{c_7^2 r^4 + 20 c_0^2} \right)^2 \tag{1.40}$$

Lemma 1.2 is proved.

It follows from Lemma 1.2 under conditions considered here a fast cycle cannot occur. This is closely related to the fact that the intersection of the spectrum of the Navier-Stokes operator with any straight line parallel to the imaginary line is limited. It would be interesting to know the conditions under which the general equation (1.5) can have a fast cycle. One of the predominant factors inducing the onset of a fast cycle is apparently the property of the spectrum of the linearized operator A to have as its limit point an infinitely distant point of the imaginary axis. It is possible that fast cycles of a certain kind occur, also, in the case of Navier-Stokes equations, but only when $\gamma_0 = \infty$, i.e. when these "branch out" from flows which remain stable at any Reynolds numbers.

The plane Couette flow, the Poiseuille flow in a round tube (although nobody has so far given a strict proof of this), and stable rotational streams [17, 18] are examples (of such flows). In our opinion it is this that explains the low stability of such modes at high Reynolds numbers, although it seems possible that at infinitely high Reynolds numbers other modes, such as stationary or almost periodic, merge with these flows.

The following analysis relates primarily to normal cycles.

Lemma 1.3. Branching out points of a cycle can only correspond to critical values of the parameter. Let $\gamma_n \to \gamma_0$ and $\omega_n \to \omega_0$ and the related nontrivial solutions $\mathbf{v}_n$ of Eq. (1.5) tends to zero by the norm of H_2. There is then among the numbers

$\mp im\omega_0$ $(m = 0, 1 \ldots)$ at least one belonging to the spectrum of operator A.

Proof. This will be made by contradiction. Let the Lemma be untrue. By assuming $\omega = \omega_0 + \mu$ we reduce Eq. (1.5) to the form

$$M\mathbf{v} \equiv \omega_0 \frac{d\mathbf{v}}{d\tau} + Av + \mu \frac{d\mathbf{v}}{d\tau} + \sum_{k=1}^{\infty} \delta^k B_k \mathbf{v} + K\mathbf{v} = 0 \tag{1.41}$$

Operator M continuously maps space $H_2 \times R \times R$ (the space of the set of three $\mathbf{v}, \mu, \delta$) into H'. For $\mathbf{v} = 0$ and $\mu = \delta = 0$ its Frechet differential is the operator L which by virtue of Lemma 1.1 is invertible. By the implicit function theorem Eq. (1.41) for sufficiently small μ and δ is only satisfied for $\mathbf{v} = 0$, which contradicts the assumption that γ_0 is a point of branching out of a cycle. The Lemma is proved.

The problem is thus reduced to finding the conditions sufficient for a given critical value of the parameter to be the point of branching out of a cycle.

2. The equation of branching out. Let us first consider the problem of determining the periodic solution of the nonhomogeneous linear equation and derive the condition of its solvability.

Lemma 2.1. Let operator A have at the imaginary axis a pair of simple eigenvalues $\mp i\omega_0$ $(\omega_0 > 0)$ with regular remaining points of that axis. For the existence of a 2π-periodic solution of equation

$$\omega_0 \frac{d\mathbf{u}}{d\tau} + A\mathbf{u} = \mathbf{f}(\tau), \qquad \mathbf{f} \in H' \tag{2.1}$$

it is then necessary and sufficient that condition

$$\int_0^{2\pi} (\mathbf{f}(\tau), \Phi)\, e^{-i\tau} d\tau = 0 \tag{2.2}$$

is satisfied.

Proof. The necessity of condition (2.2) is checked by the simple calculation: if Eq. (2.1) is solvable, then according to (2.1) and (1.10) we have

$$\int_0^{2\pi} (\mathbf{f}(\tau), \Phi)\, e^{-i\tau} d\tau = \int_0^{2\pi} (\mathbf{u}(\tau), A^*\Phi - i\omega_0\Phi)\, e^{-i\tau} d\tau = 0 \tag{2.3}$$

Let us prove the sufficiency. Let $\sigma_0(A)$, $\sigma_+(A)$ and $\sigma_-(A)$ denote the parts of spectrum $\sigma(A)$ lying on the imaginary axis within the right- and the left-hand half-planes, respectively. We denote by P_0, P_+ and P_- the related projectors

$$P_0 = \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda I - A)^{-1} d\lambda, \quad P_+ = \frac{1}{2\pi i} \int_{\Gamma_+} (\lambda I - A)^{-1} d\lambda, \quad P_- = I - P_0 - P_+ \tag{2.4}$$

Here Γ_0 and Γ_+ are smooth contours lying in the bounded part of the complex plane and consisting of regular points only of operator A. The region bounded by the contour Γ_0 (Γ_+) contains the set $\sigma_0(\sigma_+)$ and does not contain any other points of the spectrum $\sigma(A)$. The derived projectors commutate between themselves and with operator A.

We seek the solution of Eq. (2.1) in the form

$$\mathbf{u}(\tau) = \mathbf{u}_0(\tau) + \mathbf{u}_+(\tau) + \mathbf{u}_-(\tau), \quad \mathbf{u}_0 = P_0\mathbf{u}; \quad \mathbf{u}_\mp = P_\mp \mathbf{u} \tag{2.5}$$

Vector functions $\mathbf{u}_\mp$ are found from equations

$$\omega_0 \frac{d\mathbf{u}_\mp}{d\tau} + A\mathbf{u}_\mp = \mathbf{f}_\mp, \quad \mathbf{f}_\mp = P_\mp \mathbf{f} \tag{2.6}$$

It will be readily seen that these equations are solvable and

$$u_+(\tau) = \frac{1}{\omega_0} \int_{-\infty}^{\tau} e^{-\frac{1}{\omega_0}(\tau-s)A} f_+(s)\,ds, \quad u_-(\tau) = \frac{-1}{\omega_0} \int_{\tau}^{\infty} e^{-\frac{1}{\omega_0}(\tau-s)A} f_-(s)\,ds \tag{2.7}$$

The vector function u_0 is of the form

$$u_0(\tau) = \alpha(\tau)\varphi + \alpha^*(\tau)\varphi^* \tag{2.8}$$

Function $\alpha(\tau)$ satisfies equation

$$\omega_0 \frac{d\alpha}{d\tau} - i\omega_0\alpha = (f(\tau), \Phi)_H \tag{2.9}$$

The latter can be rewritten as

$$\omega_0 \frac{d}{d\tau} e^{-i\tau}\alpha(\tau) = e^{-i\tau}(f(\tau), \Phi)_H \tag{2.10}$$

Condition (2.2) implies a 2π-periodicity of function α. The following Lemma establishes the condition of solvability of Eq. (2.1).

Lemma 2.2. For a 2π-periodic solution of Eq. (2.1) to exist it is necessary and sufficient that condition

$$\int_0^{2\pi} (f(\tau), \Phi)\, e^{-in\tau}\, d\tau = 0 \tag{2.11}$$

be fulfilled for any eigenvector Φ of the conjugate operator A^* to which corresponds the eigenvalue $in\omega_0$, where n is an integer.

Proof. The proof of necessity is exactly the same as in Lemma 2.1. To prove the sufficiency we apply the Fredholm-Riesz theorem to the equivalent equation (1.18). Using relationship

$$(u, v)_{H_1} = (L_0 u, L_0 v)_{H'}, \tag{2.12}$$

we readily conclude that the related conjugate homogeneous equation is of the form

$$v + L_0^{-1} L_0^{*-1} B_0^* L_0 v = 0 \tag{2.13}$$

Operator $L_0^*: H_1 \to H'$ is defined by

$$L_0^* v \equiv -\omega_0 \frac{dv}{d\tau} + A_0 v \tag{2.14}$$

It should be noted that operator $L_0^{*-1} B_0^*$ admits absolutely continuous closure in H', since along the dense line manifold H_1' in H' it coincides with the conjugate operator $B_0 L_0^{-1} = B_0 J L_0^{-1}$ absolutely continuous in H' (see proof of Lemma 1.1).

Assuming $L_0 v = \varphi$, we see that vector function φ satisfies equation

$$-\omega_0 \frac{d\varphi}{d\tau} + A_0\varphi + B_0^*\varphi = 0 \tag{2.15}$$

If, on the other hand, φ is a 2π-periodic solution of Eq. (2.15), $v = L_0^{-1}\varphi$ satisfies Eq. (2.13).

The condition of solvability of Eq. (1.18) is of the form

$$(u_0, v)_{H_1} = (f, L_0 v)_{H'} = (f, \varphi)_{H'} = 0 \tag{2.16}$$

for any 2π-periodic solution φ of Eq. (2.15). The expansion into a Fourier series shows that the latter is a linear combination of solutions of the form $e^{in\tau}\Phi$, where n is an integer and Φ is the solution of equation

$$-in\omega_0\Phi + A^*\Phi = 0 \tag{2.17}$$

Hence condition (2.16) is equivalent to (2.11) and the Lemma is proved.

Note that the Lemma can be extended to the case in which operator B_0 is time-dependent (let us say continuously by the norm $H_1' \to H'$); this requires only the substitution of condition (2.16) for (2.11).

Let us consider Eq. (1.5) or (1.41) in which δ is a known and μ an unknown small parameter. We assume that the intersection of the spectrum of operator A with the imaginary axis consists of a pair of eigenvalues $\mp i\omega_0$ $(\omega_0 > 0)$ which we shall consider to be simple (numbers) (*).

We seek the vector function $\mathbf{v}$ in the form

$$\mathbf{v}(\tau) = \mathbf{u}(\tau) + \alpha e^{i\tau}\varphi + \alpha^* e^{-i\tau}\varphi^* \tag{2.18}$$

Constant α is uniquely defined by setting

$$\int_0^{2\pi} (\mathbf{u}(\tau), \Phi)_H e^{-i\tau}\, d\tau = 0 \tag{2.19}$$

Since Eq. (1.5) does not explicitly contain time, it has in addition to the periodic solution $\mathbf{v}$ the periodic solution $\mathbf{v}_h$ defined by $\mathbf{v}_h(\tau) = \mathbf{v}(\tau + h)$ for any real h. We define phase h by specifying a positive constant α (otherwise it would have been sufficient to pass from $\mathbf{v}$ to $\mathbf{v}_h$ with $h = -\arg \alpha$). Thus solution $\mathbf{v}$ may be sought in the form

$$\mathbf{v} = \mathbf{u} + \alpha\psi, \qquad \psi = e^{i\tau}\varphi + e^{-i\tau}\varphi^*, \qquad \alpha > 0 \tag{2.20}$$

where $\mathbf{u}$ satisfies condition (2.19). Substituting (2.20) into (1.41), we obtain

$$D\mathbf{u} \equiv \omega_0 \frac{d\mathbf{u}}{d\tau} + A\mathbf{u} = -\mu \frac{d\mathbf{u}}{d\tau} - \mu \frac{d\psi}{d\tau}\alpha - \sum_{k=1}^{\infty} \delta^k B_k(\mathbf{u} + \alpha\psi) + K(\mathbf{u} + \alpha\psi) \tag{2.21}$$

We denote by H_2° and H_0' the subspaces in H_2 and H' defined by condition (2.19) and consider D as the operator from H_2° into H_0'. By virtue of Lemma 2.1 (or 2.2) there exists the inverse operator D^{-1}.

Let us denote by P the projector in H' onto the subspace H_0'

$$P\mathbf{u} = \mathbf{u} - \varphi e^{i\tau} \frac{1}{2\pi} \int_0^{2\pi} (\mathbf{u}(s), \Phi)\, e^{-is} ds - \varphi^* e^{-i\tau} \frac{1}{2\pi} \int_0^{2\pi} (\mathbf{u}(s), \Phi^*)\, e^{is} ds \tag{2.22}$$

The problem of determining $\mathbf{u}$ and α from Eqs. (2.21) and (2.19) is equivalent to the following:

$$\mathbf{u} = D^{-1}P\left\{-\mu \frac{d\mathbf{u}}{d\tau} - \mu\alpha \frac{d\psi}{d\tau} - \sum_{k=1}^{\infty} \delta^k B_k(\mathbf{u} + \alpha\psi) + K(\mathbf{u} + \alpha\psi)\right\} \equiv D^{-1}P\mathbf{f} \tag{2.23}$$

$$(I - P)\left\{-\mu \frac{d\mathbf{u}}{d\tau} - \mu\alpha \frac{d\psi}{d\tau} - \sum_{k=1}^{\infty} \delta^k B_k(\mathbf{u} + \alpha\psi) + K(\mathbf{u} + \alpha\psi)\right\} = 0 \tag{2.24}$$

The right-hand side of Eq. (2.23) represents a continuous operator in H_2°, analytically dependent on δ, α and μ, and vanishing for $\delta = \alpha = \mu = 0$. Hence, according to the implicit function theorem, Eq. (2.23) can be solved for $\mathbf{u}$. In the neighborhood of point (0, 0, 0) this solution is analytically dependent on δ, α and μ and is uniquely

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*) It is not difficult to construct the system of equations for branching out in the general case.

defined by the condition that $\mathbf{u} = 0$ for $\delta = \alpha = \mu = 0$. It can be readily found by the method of undetermined coefficients by substituting into (2.23) expansion

$$\mathbf{u} = \sum_{k,l,m=0}^{\infty} \mathbf{u}_{klm}\delta^k\alpha^l\mu^m; \qquad u_{000} = 0 \tag{2.25}$$

We thus find that the first power terms $\mathbf{u}_{100} = \mathbf{u}_{010} = \mathbf{u}_{001} = 0$ are absent in (2.25), that $\mathbf{u}_{200} = \mathbf{u}_{101} = \mathbf{u}_{011} = \mathbf{u}_{002} = 0$, and that among the terms of second power only $\mathbf{u}_{110}$ and $\mathbf{u}_{020}$ can be nonzero

$$\mathbf{u}_{110} = -D^{-1}PB_1\psi, \qquad \mathbf{u}_{020} = D^{-1}PK\psi = D^{-1}K\phi \tag{2.26}$$

Let us write down the coefficients of the third power terms

$$\mathbf{u}_{300} = \mathbf{u}_{201} = \mathbf{u}_{102} = \mathbf{u}_{012} = \mathbf{u}_{003} = 0$$
$$\mathbf{u}_{210} = -D^{-1}P(B_1\mathbf{u}_{110} + B_2\psi), \qquad \mathbf{u}_{120} = D^{-1}P(-B_1\mathbf{u}_{020} + K_{20}{}^{\circ}(\mathbf{u}_{110}, \psi))$$
$$\mathbf{u}_{111} = -d\mathbf{u}_{110}/d\tau, \qquad \mathbf{u}_{030} = K_{20}{}^{\circ}(\mathbf{u}_{020}, \psi)$$
$$\mathbf{u}_{021} = -d\mathbf{u}_{020}/d\tau \tag{2.27}$$

As an example we also write the expression for the coefficient $\mathbf{u}_{010}$

$$\mathbf{u}_{040} = D^{-1}P(K_{20}{}^{\circ}(\mathbf{u}_{020}, \mathbf{u}_{020}) + K_{20}{}^{\circ}(\mathbf{u}_{020}, \psi)) \tag{2.28}$$

Generally for the coefficients $\mathbf{u}_{0l0}$ $(l = 3, 4, \ldots)$ we have

$$\mathbf{u}_{0l0} = D^{-1}P(K_{20}{}^{\circ}(\mathbf{u}_{0,l-1,0}, \psi) + \sum_{r=2}^{l-2} K_{20}{}^{\circ}(\mathbf{u}_{0r0}, \mathbf{u}_{0,l-r,0})) \tag{2.29}$$

We also note that $\mathbf{u}_{k0m} = 0$ $(k, m = 0, 1, \ldots)$. In fact, the sought solution of Eq. (2.23) obviously vanishes for $\alpha = 0$.

Substituting now the expansion (2.25) into (2.24), for the branching out we obtain an equation of the form

$$g(\delta, \alpha, \mu) = 0 \tag{2.30}$$

Here g is a complex-valued function, hence (2.30) is a system of two equations with two unknowns α and μ. Function g is analytic with respect to δ, α and μ in the neighborhood of point (0, 0, 0), and its expansion into a Taylor series is of the form

$$g(\delta, \alpha, \mu) \equiv \sum_{k,l,m=0}^{\infty} g_{klm}\delta^k\alpha^l\mu^m, \qquad g_{klm} = (\mathbf{f}_{klm}, \Phi e^{i\tau})_{H'} \tag{2.31}$$

where $\mathbf{f}_{klm}$ is the coefficient of $\delta^k\alpha^l\mu^m$ in the Taylor expansion of the expressions in brackets in (2.23) and (2.24). It will be readily seen that g is an odd function of the variable α: the substitution $\tau \to \tau + \pi$ and $\alpha \to -\alpha$ does not alter Eq. (2.23), while the left-hand side of Eq. (2.24) turns into its inverse number.

Taking into consideration (2.25)–(2.28), we can now write the equation of branching out (2.30) as

$$g(\delta, \alpha, \mu) \equiv -2\pi i\alpha\mu - \delta\alpha(B_1\psi, \Phi e^{i\tau})_{H'} - \delta^2\alpha(B_1\mathbf{u}_{110} + B_2\psi, \Phi e^{i\tau})_{H'} +$$
$$+ \alpha^3(K_{20}{}^{\circ}(\mathbf{u}_{020}, \psi), \Phi e^{i\tau})_{H'} + \ldots = 0 \tag{2.32}$$

where terms of fourth and higher powers of δ, α and μ have been omitted. The coefficients in Eq. (2.32) can be reduced to a simpler form by integration with respect to time. We have, for example,

$$g_{110} = -(B_1\psi, \Phi e^{i\tau})_{H'} = -\int_0^{2\pi}(B_1\varphi e^{i\tau} + B_1\varphi^* e^{i\tau}, \Phi)_H e^{-i\tau}d\tau = -2\pi(B_1\varphi, \Phi)_H \tag{2.33}$$

$$g_{210} = -2\pi(B_2\varphi, \Phi)_H + 2\pi(B_1\mathbf{W}, \Phi)_H \tag{2.34}$$

where vector $\mathbf{W}$ is the solution of equation

$$(A + i\omega_0 I)\mathbf{W} = B_1\varphi - (B_1\varphi, \Phi)_H\varphi, \quad (\mathbf{W}, \Phi)_H = 0 \tag{2.35}$$

It follows furthermore from (2.26) that the vector function $\mathbf{u}_{020}$ is of the form

$$\mathbf{u}_{020} = \mathbf{z}_0 + \mathbf{z}e^{2i\tau} + \mathbf{z}^* e^{-2i\tau} \tag{2.36}$$

where vectors $\mathbf{z}_0$ and $\mathbf{z}$ are defined by equalities

$$\mathbf{z}_0 = A^{-1}K_{20}^{\circ}(\varphi, \varphi^*) \tag{2.37}$$

$$\mathbf{z} = (A + 2i\omega_0 I)^{-1}K_{20}(\varphi, \varphi) \tag{2.38}$$

With the use of expression (2.36) we obtain

$$g_{030} = 2\pi(K_{20}^{\circ}(\mathbf{z}_0, \varphi) + K_{20}^{\circ}(\mathbf{z}, \varphi^*), \Phi)_H \tag{2.39}$$

The analysis of the branching equation makes it possible to determine the number of branching out cycles and establish their analytic properties with respect to parameter δ (for each of these α and μ, and by virtue of (2.25) also $\mathbf{u}$, are series expansions in fractional powers of parameter δ).

The following theorem specifies the conditions which make it possible to establish the existence of a cycle by an analysis of linearized equations only.

T h e o r e m 2.1. Let γ_0 be the critical value of parameter γ, and let operator A (see (1.4)) have a pair of purely imaginary simple (*) eigenvalues $\mp i\omega_0 \neq 0$.

Let there also be no eigenvalues of operator A among the numbers $in\omega$ (n is an integer and $n \neq \mp 1$), and let the following condition be satisfied:

$$\operatorname{Re}(B_1\varphi, \Phi)_H \neq 0 \tag{2.40}$$

Then γ_0 is the point of the branching out of a cycle. There can be only two possibilities in this case: either we have a single normal one-sided cycle, or Eq. (1.5) has for $\gamma = \gamma_0$ and $\omega = \omega_0$ a one-parametric set of 2π-periodic solutions $\{\mathbf{v}_\alpha\}$ with $\mathbf{v}_0 = 0$ analytically dependent on the small parameter α, and, if at the same time $\gamma - \gamma_0$ is fairly small but not zero, Eq. (1.5) has no small periodic solutions.

P r o o f. By cancelling in (2.32) $2\pi\alpha$, we obtain equation

$$h(\delta, \alpha^2, \mu) \equiv i\mu + (B_1\varphi, \Phi)_H\delta + \ldots = 0 \tag{2.41}$$

where terms containing δ, α and μ of powers higher than the first have been omitted.

Assuming $\operatorname{Re} h = h$ and $\operatorname{Im} h = h_i$ and calculating the Jacobian, we obtain

$$\left.\frac{\partial(h_r, h_i)}{\partial(\delta, \mu)}\right|_{\alpha=\delta=\mu=0} = \operatorname{Im}\left.\frac{\partial h}{\partial\mu}\frac{\partial h^*}{\partial\delta}\right|_{\alpha=\delta=\mu=0} = \operatorname{Re}(B_1\varphi, \Phi)_H \neq 0 \tag{2.42}$$

---

*) We recall that this implies not only uniqueness of eigenvectors φ and φ^* corresponding to eigenvalues $-i\omega_0$ and $i\omega_0$ but, also, the absence of adjoint vectors, which leads to the condition $(\varphi, \Phi)_H \neq 0$ (see (1.11)).

Using the implicit function theorem, we conclude from this that Eq. (2.41) can for sufficiently small $|\alpha|$ be solved for δ and μ. since there exist analytic functions $\delta = \theta(\alpha^2)$ and $\mu = \rho(\alpha^2)$ which reduce (2.41) to an identity and are uniquely defined by the stipulation that $\theta(0) = \rho(0) = 0$.

Let us take an arbitrary sequence of positive numbers $\alpha_n \to 0$ and construct the corresponding sequencies

$$\delta_n = \theta(\alpha_n^2) \to 0, \qquad \mu_n = \rho(\alpha_n^2) \to 0, \qquad \omega_n = \omega_0 + \mu_n$$

Then the series (2.25) yield the related solution of Eq. (2.21), and formula (2.20) that of Eq. (1.5) which is obviously nontrivial, since $\alpha_n > 0$. This proves that γ_0 is a point of branching out of a cycle.

If the left-hand side of Eq. (2.41) is independent of α, then the unique small solution of that equation is obviously $\mu = \delta = 0$. In this case parameter α remains arbitrary and, if it is sufficiently small, the series (2.25) is a solution of Eq. (2.23) for $\delta = \mu = 0$. Thus in this case the second of the possibilities noted in Theorem 2.1 is realized (*).

Let now the left-hand side of Eq. (2.41) depend on α. We expand function θ into a Taylor series and try to solve the equation

$$\delta = \theta(\alpha^2) = c_m\alpha^{2m} + c_{m+1}\alpha^{2(m+1)} + \ldots \tag{2.43}$$

for α at small δ.

We assume that $c_m \neq 0$ and $m \geqslant 1$. If $c_m > 0$ ($c_m < 0$), Eq. (2.43) has a unique small positive root α for $\delta > 0$ ($\delta < 0$) and has no small real roots for $\delta < 0$ ($\delta > 0$). In both cases we have

$$\alpha = \left(\frac{\delta}{c_m}\right)^{1/2m}\left[1 - \frac{c_{m+1}}{2mc_m}\left(\frac{\delta}{c_m}\right)^{1/m} + \ldots\right] \tag{2.44}$$

The square brackets contain the series expansion in powers of parameter $(\delta/c_m)^{1/m}$.

To prove the expression (2.44) it is sufficient to apply the implicit function theorem to the equation derived from (2.43) by dividing it by c_m and extracting a $2m$-th power root from both of its sides.

Thus the uniqueness and one-sidedness of the cycle are established for this case. Since $\omega_n = \omega_0 + \mu_n \to \omega_0$, this cycle is obviously normal. Theorem 2.1 is proved.

A more detailed investigation into the existence of small cycles, their number, and analytic properties must take into consideration the nonlinear terms. A typical and one of the simplest cases are described by statement that follows.

T h e o r e m 2.2. Let all conditions of Theorem 2.1 and the inequality

$$\operatorname{Re} g_{030} \equiv \operatorname{Re}(K_{20}^{\circ}(u_{020}, \psi), \Phi e^{i\tau})_{H'} \neq 0 \tag{2.45}$$

be satisfied. Then γ_0 is the point of branching out of a single one-sided cycle which exists for small $\delta > 0$ ($\delta < 0$), if $\operatorname{Re} g_{030} / \operatorname{Re} g_{110} < 0$ (> 0), and is an analytic function of parameter $\sqrt{\delta}$ ($\sqrt{-\delta}$), and μ is an analytic function of δ.

We then have

$$\alpha = \left(-\frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{030}}\delta\right)^{1/2} + O(\delta)$$

$$\mu = \frac{1}{2\pi}\delta\left(\operatorname{Im} g_{110} - \operatorname{Re} g_{110}\frac{\operatorname{Im} g_{030}}{\operatorname{Re} g_{030}}\right) + O(\delta^2) \tag{2.46}$$

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*) This case is obviously exceptional. It can, for example, occur when $K\psi = 0$. Whether this is possible in the case of Navier-Stokes equations is not known. It is not difficult, however, to give examples of nonlinear operators with this property.

Magnitudes g_{110} and g_{030} are defined by equalities (2.33) and (2.39).

Proof. Reverting to Eq. (2.41) and writing the expressions for terms containing δ, α^2 and μ of the first power in an explicit form, we obtain

$$h(\delta,\ \alpha^2,\ \mu) \equiv i\mu - \frac{1}{2\pi}\ g_{110}\,\delta - \frac{1}{2\pi}\ g_{030}\,\alpha^2 + \ldots = 0 \qquad (2.47)$$

Equation (2.47) can be solved for α^2 and μ, since there exist functions $\alpha^2 = \xi(\delta)$ and $\mu = \eta(\delta)$ which are analytic at point $\delta = 0$, reduce Eq. (2.47) to an identity, and are uniquely defined by the stipulation that $\xi(0) = \eta(0) = 0$. This follows from the implicit function theorem, since condition (2.45) implies the inequality

$$\left.\frac{\partial (h_r,\ h_i)}{\partial (\alpha^2,\ \mu)}\right|_{\alpha=\delta=0} = \operatorname{Im} \left.\frac{\partial h}{\partial \mu}\,\frac{\partial h^*}{\partial \alpha^2}\right|_{\alpha=\delta=0} = \operatorname{Im} i g_{030} = \operatorname{Re} g_{030} \neq 0 \qquad (2.48)$$

From (2.48) we further have

$$\xi'(0) = -\frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{030}}, \quad \eta'(0) = \frac{1}{2\pi}\operatorname{Im} g_{110} + \frac{1}{2\pi}\ \operatorname{Im} g_{030}\,\xi'(0)$$

from which immediately follow the relationships (2.46). Thus Theorem 2.2 is proved.

The branching equation (2.32) can be readily analyzed for various exceptional cases. In fact, according to the implicit function theorem, μ appearing in equation $h_i\ (\delta,\ \alpha^2,\ \mu) = 0$ can be expressed in terms of a series expansion in powers of δ and α^2, since $(\partial/\partial\mu)h_i\ (0,\ 0,\ 0) = 1 \neq 0$. Substituting this expansion into equation $h_r\ (\delta,\ \alpha^2,\ \mu) = 0$, we obtain equation $f\ (\delta,\ \alpha^2) = 0$. A complete analysis of the latter can be made with the use of the Newton diagram. For example, it can be readily shown in this way that under conditions of Theorem 2.1 with $\operatorname{Re}\ g_{030} = 0$, but

$$\operatorname{Re} g_{050} \equiv \operatorname{Re} (K_{20}{}^\circ (\mathbf{u}_{040},\ \psi) + 2K_{20}{}^\circ (\mathbf{u}_{020},\ \mathbf{u}_{030}),\ \Phi e^{i\tau})_{H'} \neq 0 \qquad (2.49)$$

a single normal one-sided cycle analytically dependent on $\delta^{1/4}$ or $(-\delta)^{1/4}$ branches out, while μ is analytically dependent on $\delta^{1/2}$ or $(-\delta)^{1/2}$ and

$$\alpha = \left(-\frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{050}}\,\delta\right)^{1/4} + O\left(\delta^{3/4}\right) \qquad (2.50)$$

$$\mu = \frac{1}{2\pi}\operatorname{Im} g_{030}\left(-\frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{050}}\,\delta\right)^{1/2} + \frac{1}{2\pi}\,\delta\left(\operatorname{Im} g_{110} - \frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{050}}\operatorname{Im} g_{050}\right) + O\left(\delta^{3/2}\right)$$

Let us also consider the case when $\operatorname{Re} g_{110} = 0$ and condition (2.40) are not satisfied. We eliminate μ from Eq. (2.47) by substituting for it its expression in the form of a series expansion in powers of δ and α^2

$$\mu = \frac{1}{2\pi}\operatorname{Im} g_{110}\,\delta + \frac{1}{2\pi}\operatorname{Im} g_{030}\alpha^2 + \ldots \qquad (2.51)$$

where terms containing δ and α^2 of powers higher than the first have been omitted. For the determination of α we obtain equation

$$\operatorname{Re} g_{030}\alpha^2 + \operatorname{Re} g_{210}\delta^2 + \ldots = 0 \qquad (2.52)$$

where terms containing δ and α of the third and higher powers have been omitted. It is now obvious that there are no small cycles when

$$\operatorname{Re} g_{030} \operatorname{Re} g_{210} > 0 \qquad (2.53)$$

This completely explains the role of condition (2.40) in Theorem 2.1. With all other conditions of the theorem satisfied, this condition is necessary and sufficient for the critical value γ_0 to be the point of branching out of a cycle for "arbitrary" purely nonlinear operator K and linear operators B_1 and B_2.

If instead of (2.53) the opposite inequality

$$\operatorname{Re} g_{030} \operatorname{Re} g_{210} < 0 \tag{2.54}$$

is valid, then it follows directly from (2.52) that there exists a unique normal two-sided cycle for which

$$\alpha = \left(-\frac{\operatorname{Re} g_{210}}{\operatorname{Re} g_{030}}\right)^{1/2} \delta + O(\delta^2) \tag{2.55}$$

3. Extension. In the foregoing analysis the properties of the Navier-Stokes equations were used to a small extent only, hence an extension to a fairly wide class of ordinary differential equations in a Banach space is not difficult. This class of equations includes numerous problems of mathematical physics such as, for example, nonlinear parabolic equations, equations of magnetohydrodynamics, etc. We would note that the assumption of analyticity with respect to parameter δ, as well as that introduced below on the analyticity with respect to u, need not be strictly adhered to: it is sufficient to specify only a few continuous derivatives (in the case of Theorems 2.1 and 3.1 only the first derivatives need be continuous).

Let us derive the nontrivial 2π-periodic solutions of the ordinary differential equation

$$\omega \frac{dv}{d\tau} + Av = K(v, \delta) \tag{3.1}$$

in the Banach space X on the following assumptions.

1) A is a linear operator generating the operator of the semigroup. The intersection of its spectrum with the imaginary axis consists of a pair of simple poles $\mp i\omega_0 \neq 0$.

We retain the previously used definitions (1.6) and (1.10) for the eigenvectors of operators A and A^*.

Let W_p be the Banach space of the 2π-periodic vector function of parameter τ whose values in X have the finite norm

$$\|v\|_{W_p} = \left\{\int_0^{2\pi} \left[\left\|\frac{dv(\tau)}{d\tau}\right\|_X^p + \|Av(\tau)\|_X^p\right] d\tau\right\}^{\frac{1}{p}} + \max_\tau \|v(\tau)\|_X \tag{3.2}$$

Here $p > 1$ is a certain number.

2) We assume that for any $\omega \mp \omega_0$ and $f \in L_p([0, 2\pi], X)$ equation

$$\omega \frac{du}{dt} + Au = f \tag{3.3}$$

has a unique 2π-periodic solution $u = L_\omega f$, and that operator L_ω continuously acts from $L_p([0, 2\pi], X)$ into W_p (coercivity).

3) For any sufficiently small δ the nonlinear operator K acts absolutely continuously from W_p into $L_p([0, 2\pi], X)$ and in the vicinity of the zero of space $X \times R$ is analytic with respect to the set v, δ. Let its expansion into a Taylor series be of the form

$$K(v, \delta) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} K_{mn} v^m \delta^n, \qquad K_{10} = 0 \tag{3.4}$$

The following notation is used here:

$$K_{mn} v^m = K_{mn}(v, v, \ldots, v) \tag{3.5}$$

where K_{mn} is a linear operator with respect to each of its arguments.

Let us set

$$K_{1n} = -B_n \ (n = 1, 2, \ldots)$$

In the case of the Navier-Stokes equations considered above $X = H$; $p = 2$; $W_p = H_2$; B_m and K_{20} are defined by equalities (1.4), and the remaining operators K_{mn} are equal zero.

Conditions 1–3 make it possible to extend the previously derived results to the general case. Since the proofs remain the same, the subject reduces to a branching equation of the form (2.30). Hence we confine ourselves to the formulation of the theorem.

Theorem 3.2. Let conditions 1–3 and

$$\operatorname{Re}(B_1\ \varphi,\ \Phi) \neq 0 \tag{3.6}$$

be satisfied. The $\delta = 0$ is the point of branching out of a cycle, and we have either a unique normal one-sided cycle, or for $\delta = 0$ and $\omega = \omega_0$ Eq. (3.1) has a set of 2π-periodic solutions $\{v_\alpha\}$ with $v_0' = 0$, while for small $\delta \neq 0$ there are no nontrivial 2π-periodic solutions (*).

Theorem 3.2. Let the conditions of Theorem 3.1 and the inequality

$$\operatorname{Re} g_{030} = \operatorname{Re}(K_{20}^{\circ}(\varphi, z_0) + K_{20}^{\circ}(\varphi^*, z) + K_{30}^{\circ}(\varphi, \varphi, \varphi^*), \Phi) \neq 0 \tag{3.7}$$
$$K_{30}^{\circ}(u, u, v) = K_{30}(v, u, u) + K_{30}(u, v, u) + K_{30}(u, u, v)$$

be satisfied. Then $\delta = 0$ is the point of branching out of a unique normal one-sided cycle which is an analytic function of parameter $\sqrt{\delta}$ or $\sqrt{-\delta}$, and $\mu = \omega - \omega_0$ is an analytic function of δ. In this case

$$v = \left(-\frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{030}}\delta\right)^{1/2}\psi + O(\delta)$$
$$\mu = \frac{1}{2\pi}\delta\left(\operatorname{Im} g_{110} - \operatorname{Re} g_{110}\frac{\operatorname{Im} g_{030}}{\operatorname{Re} g_{030}}\right) + O(\delta^2) \tag{3.8}$$

The conclusions presented at the end of the preceding Section are, also, valid in the general case, although the expressions for coefficients g_{klm} become somewhat more involved.

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*) The theorem is also valid, when in addition to $\mp i\omega_0$ other points of the spectrum lie on the imaginary axis, provided however there are no $in\omega_0$ ($n = 0, \mp 2, \mp 3, \ldots$). among these. It should be noted, however, that the statements about uniqueness and non-existence do not relate then to any arbitrary cycles but to such in which $\omega \to \omega_0$ for $\delta \to 0$. If there are several pairs of purely imaginary eigenvalues $\mp \omega_{01}, \mp \omega_{02}, \ldots, \mp i\omega_{0p}$ and the numbers ω_{0j} and ω_{0k} for $k \neq j$ are incommesurable, then Theorems 3.1 and 3.2 yield conditions for the onset of p cycles.

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